Distributional solution concepts for the Euler-Bernoulli beam equation with discontinuous coefficients*

Günther Hörmann
Fakultät für Mathematik der Universität Wien
Nordbergstraße 15, A-1090 Wien, Austria

and

Ljubica Oparnica
Institute of Mathematics, Serbian Academy of Science
Kneza Mihajla 35, 11000 Belgrade, Serbia

April 2007

Abstract

We study existence and uniqueness of distributional solutions w to the ordinary differential equation $\frac{d^2}{dx^2}\left(a(x) \cdot \frac{d^2 w(x)}{dx^2}\right) + P(x)\frac{d^2 w(x)}{dx^2} = g(x)$ with discontinuous coefficients and right-hand side. For example, if a and w are non-smooth the product $a \cdot w''$ has no obvious meaning. When interpreted on the most general level of the hierarchy of distributional products discussed in [10, Chapter II], it turns out that existence of a solution w forces it to be at least continuously differentiable. Curiously, the choice of the distributional product concept is thus incompatible with the possibility of having a discontinuous displacement function as a solution. We also give conditions for unique solvability.

Key words: ordinary differential equations with discontinuous coefficients, distributional solutions, multiplication of distributions.
AMS 2000 Subject Classification: 46F10;34A36.

1   Equation of the Euler-Bernoulli beam

We consider an Euler Bernoulli rod under a distributed transversal force g and axial force P. The differential equation of equilibrium for the displacement w is given in [1] in the form

$$\frac{d^2}{dx^2}\left(EI\frac{d^2 w(x)}{dx^2}\right) + P\frac{d^2 w(x)}{dx^2} = g(x) \qquad x \in [0, l]. \tag{1}$$

Here, E is the modulus of elasticity, I is the moment of inertia, and l is the length of the rod. In our analysis we will allow for nonconstant, x-depended, even discontinuous coefficients I and P. When there is a discontinuity in I at some point x_0 the rod can be considered to consist of two different, but connected, parts, i.e., $EI(x) = EI_1 + H(x - x_0)(EI_2 - EI_1)$ where $I_1 \neq I_2$ are the corresponding moments of inertia respectively and H denotes the Heaviside function.

Equation (1) has been studied in [12], where the authors discuss possible jump discontinuities at x_0 in the displacement

$$\Delta := w(x_0+) - w(x_0-) \tag{2}$$

(as well as in the rotation $\theta := w'(x_0+) - w'(x_0-)$), where a suffix $+$ or $-$ in the function argument denotes the limit from the right or left. A solution Ansatz of the form $w = w_1 + H(x - x_0)(w_2 - w_1)$ is then used, where w_1 and w_2 solve the equation to the left and to the right of x_0. In course of justifying this could be called a solution Δ was being forced to vanish in order to avoid ill-defined products involving a Dirac delta.

Here, we investigate the corresponding mathematical issues left open: first, we analyze the possibility to give a meaning to the notion of 'distributional solution' in the context of the distributional product hierarchy described in [10, Chapter II] (see also the Appendix for a brief review); second, we show that indeed Δ necessarily has to vanish then, which is consistent with the calculations in [12]; more precisely, if w were to have a jump discontinuity then the model product $[a \cdot w'']$, which is the most general in the distributional product hierarchy, can not exist. Thus, in order to allow for solutions with jump discontinuities in the displacement one is forced to go beyond intrinsic distributional products and use, e.g. algebras of generalized functions (cf. [2, 10]). For example, there has been active research on such issues for hyperbolic partial differential equations with discontinuous coefficients, where in certain cases non-existence of distributional solutions has been proved (cf. [4, 5, 6, 7, 8, 9]).

For notational simplification and structural clarity we put $A = EI_1$, $B = EI_2$ (hence $A \neq B$) and

$$a(x) = A + (B - A)H(x - x_0) = AH(x_0 - x) + BH(x - x_0). \tag{3}$$

Then the governing differential equation with boundary conditions for the Euler-Bernoulli rod with jump discontinuities in the bending read

$$\frac{d^2}{dx^2}\left(a(x) \cdot \frac{d^2 w(x)}{dx^2}\right) + P(x)\frac{d^2 w(x)}{dx^2} = g(x), \qquad x \in [0, 1] \tag{4}$$

$$w(0) = 0; \qquad w(l) = 0;$$
$$\frac{d^2 w}{dx^2}(0) = 0; \qquad \frac{d^2 w}{dx^2}(l) = 0. \tag{5}$$

Mechanically, a global condition of equilibrium is expressed by equality of the bending moments

$$EI_1 w''(x_0-) = EI_2 w''(x_0+). \tag{6}$$

We may use the substitution $u = w''$ to lower the order of equation (4) and boundary conditions (5)

$$\frac{d^2}{dx^2}\left(a(x) \cdot u(x)\right) + P(x)\,u(x) = g(x) \qquad x \in [0, 1] \tag{7}$$

with

$$u(0) = u(1) = 0. \tag{8}$$

Remark 1.1. Note that the above substitution is equivalent to imposing the additional boundary problem $\frac{d^2}{dx^2}w(x) = u(x)$ with $w(0) = w(1) = 0$, which is uniquely solvabile once u is determined by (7-8). In the sequel we will thus only consider u.

In equation (7) the product of the distributions a and u arises. There are several concepts of partialy defined products in the space of distributions. In the current paper we use the so-called model product (cf. [10]) to give a meaning to the differential equation.

Remark 1.2. (Comparison with L^2-operator theory.) The above boundary value problem (7-8) can as well be investigated in the classical functional analytic context of unbounded operators on $L^2([0, 1])$. Singularities of the coefficient functions then have a significant influence on choices for an appropriate domain. In course of the current paper, we follow an intrinsic distribution theoretic view, which allows for a wider class of solutions, right-hand sides in the differential equation, as well as variations in the solution concept itself.

To illustrate the situation in an unbounded operator approach we briefly sketch the constructions for the case where a is given by (3) and P is a real constant. It is natural to implement the boundary conditions (8) into the domain of the operator. Furthermore, we have to specify the meaning of the formal expression $(au)''$. Note that requiring that u belongs to the Sobolev space $H^2(]0,1[)$ makes u'' well-defined in $L^2([0,1])$ and gives sense to the boundary conditions $u(0) = u(1) = 0$. Observe that under these hypotheses $au = Au_- + Bu_+$, where u_- (resp. u_+) vanishes to the right (resp. left) of x_0 and is continuously differentiable on the left (resp. right) up to x_0. Thus, by Schwartz' formula ([11, Chapitre II, §2]), we have $(au)'' = au'' + (Bu'_+(x_0) - Au'_-(x_0)) \cdot \delta_{x_0} + (Bu_+(x_0) - Au_-(x_0)) \cdot \delta'_{x_0}$, which is in L^2 only for u such that the coefficients of δ_{x_0} and δ'_{x_0} vanish.

Therefore, we define the operator $Tu := a \cdot u''$ with domain

$$D(T) := \{u \in H^2(]0,1[) : u(0) = u(1) = 0, Bu_-(x_0) - Au_+(x_0) = 0, Bu'_+(x_0) - Au'_-(x_0) = 0\}.$$

It is straightforward to check that T is symmetric, i.e., $D(T) \subseteq D(T^*)$ and $T^* \mid_{D(T)} = T$, where T^* denotes the adjoint of T. In fact, one can prove that T is self-adjoint along the following lines: Let $\varphi \in \mathcal{C}_c^\infty(]0,1[) \cap D(T)$ and $v \in D(T^*)$; interpreting L^2-inner products $\langle . | . \rangle$ in terms of distributional actions $\langle . , . \rangle$ and vice versa we obtain $\overline{\langle T^*v, \varphi \rangle} = \langle \varphi | T^*v \rangle = \langle T\varphi | v \rangle = \langle a\varphi'' | v \rangle = \overline{\langle (av)'', \varphi \rangle}$, which implies that $(av)'' \in L^2$, forcing that v belongs to H^2 and satisfies the conditions appearing in $D(T)$ at x_0. Furthermore, integration by parts is then applicable with $u \in D(T)$ yielding $\langle Tu | v \rangle = \langle u | a \cdot v'' \rangle + Bu'(1)v(1) - Au'(0)v(0)$; since $u \mapsto \langle Tu | v \rangle$ has to be a continuous linear functional with respect to the L^2-norm $v(0)$ and $v(1)$ have to vanish. Hence v is in $D(T)$ and $T^* = T$.

We observe that the original differential operator in Equation (7) is of the form $T + PI$, where I denotes the identity operator. Therefore, questions concerning uniqueness and existence of solutions to (7-8) when $g \in L^2$ directly relate to spectral properties of T. One can view corresponding results obtained in Section 3 below in this context.

2 Solution concept based on the model product

We analyze the properties of a distributional solution u to problem (7-8) in detail when the product $a \cdot u$ is interpreted as a 'model product'. Throughout this and the following two sections we focus on regularity issues stemming from the highest order terms in the equation. Therefore we make the assumption that

$$P \text{ is constant.}$$

We will remove this assumption and generalize our results in a final section allowing for jump discontinuities in P as well.

Definition 2.1. Let $\dot{\mathcal{D}}'([0,1]) := \{v \in \mathcal{D}'(\mathbb{R}); \operatorname{supp} v \subseteq [0,1]\}$. We call $u \in \dot{\mathcal{D}}'([0,1])$ a solution to (7-8) if the following holds:

(A1) The model product $[a \cdot u]$ of u and a (defined as in [10], see also the Appendix) exists in $\mathcal{D}'(\mathbb{R})$

(A2) The equation

$$([a \cdot u])'' + Pu = g \tag{9}$$

holds in $\mathcal{D}'(\mathbb{R})$.

Remark 2.2. (i) The boundary conditions (8) are implemented into the definition of the space of prospective solutions $\dot{\mathcal{D}}'([0,1])$ in the following sense: if $u \in \dot{\mathcal{D}}'([0,1])$ happens to be a continuous function then $u(0) = u(1) = 0$.

(ii) Note that (A1) is equivalent to the existence of the model products $[H_- \cdot u]$ and $[H_+ \cdot u]$ where $H_-(x) = H(x_0 - x)$ and $H_+ = H(x - x_0)$.

Lemma 2.3. *(i) Let $u \in \dot{\mathcal{D}}'([0,1])$ satisfy (A1-2) then $[H_- \cdot u]$ and $[H_+ \cdot u]$ belong to $\dot{\mathcal{D}}'([0,1])$.*
(ii) $[H_- \cdot \delta_{x_0}^{(k)}]$ and $[H_+ \cdot \delta_{x_0}^{(k)}]$ exist if and only if $k = 0$, in which case we have $[H_- \cdot \delta_{x_0}] = -\frac{\delta_{x_0}}{2}$, $[H_+ \cdot \delta_{x_0}] = \frac{\delta_{x_0}}{2}$. (Cf. similar investigations in [5, Lemma 4])

Proof. Let $\varphi \in \mathcal{D}(\mathbb{R})$ with $\int \varphi = 1$ and $\varphi_\varepsilon(x) := \varphi(x/\varepsilon)/\varepsilon$ be a model delta net (cf. [10],(7.9)).
(i) By definition $[H_\pm \cdot u] = \lim_{\varepsilon \to 0}(H_\pm * \varphi_\varepsilon) \cdot (u * \varphi_\varepsilon)$. Let $w_\varepsilon := (H_\pm * \varphi_\varepsilon) \cdot (u * \varphi_\varepsilon)$ and $\psi \in \mathcal{D}(\mathbb{R})$ with $\operatorname{supp}\psi \cap [0,1] = \emptyset$ then $\langle w_\varepsilon, \psi \rangle = \langle H_\pm * \varphi_\varepsilon, (u * \varphi_\varepsilon) \cdot \psi \rangle$. Since $\operatorname{supp}(u * \varphi_\varepsilon) \subseteq [0,1] + \operatorname{supp}\varphi_\varepsilon \subseteq [-d_\varepsilon, 1 + d_\varepsilon]$ for some $d_\varepsilon \to 0$ (as $\varepsilon \to 0$) we have $(u * \varphi_\varepsilon) \cdot \psi = 0$ and thus $\langle w_\varepsilon, \varphi \rangle = 0$.
(ii) For any $\psi \in \mathcal{D}(\mathbb{R})$

$$\langle [H_- \cdot \delta_{x_0}^{(k)}], \psi \rangle = \lim_{\varepsilon \to 0} \langle (H(x_0 - x) * \varphi_\varepsilon)(\delta_{x_0}^{(k)} * \varphi_\varepsilon), \psi \rangle$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{k+1}} \int_{\mathbb{R}} \int_{\frac{x - x_0}{\varepsilon}}^{\infty} \varphi(t)\varphi^{(k)}(\frac{x - x_0}{\varepsilon})\psi(x)\,dt dx$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^k} \int_{\mathbb{R}} \int_{z}^{\infty} \varphi(t)\varphi^{(k)}(z)\psi(\varepsilon z + x_0)\,dt dz$$

cannot be convergent for all ψ as $\varepsilon \to 0$ if $k \neq 0$. In case $k = 0$ we obtain the formula $[H_- \cdot \delta_{x_0}] = -\delta_{x_0}/2$ by dominated convergence and the fact that $\int_{\mathbb{R}} \int_z^\infty \varphi(t)\varphi(z)\,dz dt = -1/2$. The proof for $[H_+ \cdot \delta_{x_0}]$ is similar. $\blacksquare$

Theorem 2.4. *Let $u \in \dot{\mathcal{D}}'([0,1])$ be a solution in the sense of Definition 2.1. Then u is a locally integrable function.*

Proof. *Step 1:* Putting $\widetilde{u}_- = u\,|_{(0,x_0)}$ and $\widetilde{u}_+ = u\,|_{(x_0,1)}$ yields

$$A\widetilde{u}''_- + P\widetilde{u}_- = g\,|_{(0,x_0)} \tag{10}$$
$$B\widetilde{u}''_+ + P\widetilde{u}_+ = g\,|_{(x_0,1)}. \tag{11}$$

Solving these two differential equations with constant coefficients we get

$$\widetilde{u}_- = \widetilde{u}_{-h} + \widetilde{u}_{-p} \qquad \text{and} \qquad \widetilde{u}_+ = \widetilde{u}_{+h} + \widetilde{u}_{+p}, \tag{12}$$

where

$$\widetilde{u}_{-h}(x) = C_1 e^{\sqrt{-P/A}x} + C_2 e^{-\sqrt{-P/A}x}, \quad \widetilde{u}_{+h}(x) = D_1 e^{\sqrt{-P/B}x} + D_2 e^{-\sqrt{-P/B}x}$$

and

$$\widetilde{u}_{-p}(x) = \frac{1}{2\sqrt{-P/A}}\left(\int_0^x g(\tau)e^{\sqrt{-P/A}(x-\tau)}d\tau - \int_0^x g(\tau)e^{-\sqrt{-P/A}(x-\tau)}d\tau\right),$$

with a similar formula for $\widetilde{u}_{+p}(x)$ replacing P/A by P/B and integration limits from 1 to x. Here, $\widetilde{u}_{-h}$, $\widetilde{u}_{+h}$ are smooth and $\widetilde{u}_{-p}$, $\widetilde{u}_{+p}$ are absolutely continuous. Therefore $\widetilde{u}_-$ and $\widetilde{u}_+$ are absolutely continuous functions on open subintervals $(0, x_0)$ and $(x_0, 1)$ respectively. Also, by explicit formula, we see that $\widetilde{u}_-(x_0-) := \lim_{x \to x_0-} \widetilde{u}_-(x)$ and $\widetilde{u}_+(x_0+) := \lim_{x \to x_0+} \widetilde{u}_+(x)$ exist.
Step 2: Define $\widetilde{u} \in L^1_{\text{loc}}(\mathbb{R})$ by

$$\widetilde{u}(x) = \begin{cases} 0 & -\infty < x \leq 0 \\ \widetilde{u}_-(x) & 0 < x < x_0 \\ \widetilde{u}_+(x) & x_0 < x < 1 \\ 0 & 1 \leq x < \infty \end{cases}. \tag{13}$$

4

We have that $\widetilde{u} \in \dot{\mathcal{D}}'([0,1])$ and $(u - \widetilde{u})\mid_{\mathbb{R}\setminus\{x_0\}} = 0$. Therefore $\mathrm{supp}(u - \widetilde{u}) = \{x_0\}$, which implies that

$$u = \widetilde{u} + \sum_{k=0}^{N} c_k \delta_{x_0}^{(k)}, \qquad c_k \in \mathbb{C}, N \in \mathbb{N}_0. \tag{14}$$

By Lemma 2.3 and Assumption (A1) $N = 0$ in (14). Hence

$$u = \widetilde{u} + c_0 \delta_{x_0}. \tag{15}$$

Step 3: By Assumption (A2) we now obtain

$$(u \cdot a)'' = g - Pu = g - P\widetilde{u} - cP\delta_{x_0}, \tag{16}$$

where $g - P\widetilde{u} \in L_{\mathrm{loc}}^1(\mathbb{R})$. Let w be a primitive function for $g - P\widetilde{u}$. Then $w - c_0 PH(x - x_0)$ is one for $(u \cdot a)'$. Therefore

$$u \cdot a = W - c_0 P(x - x_0)_+,$$

where x_+ denote kink function, i.e. $x_+ = \begin{cases} x & x > 0 \\ 0 & x \leq 0 \end{cases}$ and W is primitive function for w. Since $W \in C^1$ and the kink function is absolutely continuous we have that $u \cdot a$ is absolutely continuous. But then (15) and (3) imply that

$$u \cdot a = \widetilde{u} \cdot a + \frac{c_0}{2} \delta_{x_0}(B - A)$$

which is absolutely continuous if and only if $c_0 = 0$. This in turn yields $u = \widetilde{u}$ and therefore u is locally integrable as $\widetilde{u}$ is. ∎

3 Existence and uniqueness of an $L^1([0,1])$-solution

As we have seen in the previous section, a distributional solution in the sense of Definition 2.1 necessarily is a locally integrable function. In this case, we can interpret the product $u \cdot a$ as a duality product (cf. [10] or the Appendix). We analyze this situation more closely.

Proposition 3.1. *If $u \in L^1([0,1])$ is a solution to (7-8) then $u \in C^1([0,1]\setminus\{x_0\})$ and u has a jump at $x = x_0$.*

Proof. If $u \in L^1([0,1])$ then the differential equation (7) yields $(u \cdot a)'' \in L^1([0,1])$, hence $(u \cdot a)' \in C_{\mathrm{abs}}([0,1])$ and thus $u \cdot a \in C^1([0,1])$. Therefore we also have that $u \cdot a \mid_{[0,x_0)} = Au \mid_{[0,x_0)} \in C^1([0,x_0))$ in turn $u \in C^1([0,x_0))$. Similarly, $u \in C^1((x_0,1])$. Furthermore $\lim_{x \to x_0^-} u \cdot a(x) = A \cdot u(x_0)$ and therefore $u(x_0-) = \lim_{x \to x_0^-} u(x)$ exists. Similarly for $u(x_0+) = \lim_{x \to x_0^+} u(x)$. But $a \cdot u$ is continuous, so that $\lim_{x \to x_0^-} u \cdot a(x) = \lim_{x \to x_0^+} u \cdot a(x)$ and thus

$$Au(x_0-) = Bu(x_0+), \tag{17}$$

which implies the global equilibrium condition (6). If $u \in C([0,1])$ then (17) implies $A = B$, which contradicts the assumption $I_1 \neq I_2$. This means that u has to be discontinuous at x_0. ∎

Remark 3.2. As a matter of fact we have $u' \in C_{\mathrm{abs}}([0,1]\setminus\{x_0\})$. Indeed, since $(u \cdot a)' \in C_{\mathrm{abs}}([0,1])$ reasoning as above we obtain that u' is absolutely continuous off x_0, so that $u'(x_0-)$, $u'(x_0+)$ exist and obtain

$$Au'(x_0-) = Bu'(x_0+). \tag{18}$$

Now we are in a position to construct a solution to (7-8).

Lemma 3.3. *For any choice of $A > 0$, $B > 0$, and $0 < x_0 < 1$ there exists a strictly increasing sequence $(P_l)_{l \in \mathbb{N}}$ of positive real numbers P_l such that the following holds:*

(i) If $P < 0$

or

(ii) if $P > 0$ and $P \neq P_l$ for all $l \in \mathbb{N}$

then there is a unique solution to (10) and (11) with $\widetilde{u}_-(0) = 0$ and $\widetilde{u}_+(1) = 0$, which satisfies the stability conditions

$$A\widetilde{u}_-(x_0) = B\widetilde{u}_+(x_0) \tag{19}$$
$$A\widetilde{u}'_-(x_0) = B\widetilde{u}'_+(x_0). \tag{20}$$

Remark 3.4. In case $P = P_l$ for some $l \in \mathbb{N}$ the solution is not unique or even may fail to exist. Investigation of these cases seems possible in a direct way without requiring further analytical tools.

Proof. Any solutions to (10) and (11) are given by (12).

Case $P < 0$: The solution formulae (12), adapted to the boundary conditions at 0 and 1, give

$$\widetilde{u}_-(x) = 2C_1 \sinh \sqrt{-P/A}\, x + \widetilde{u}_{-p}(x)$$

and

$$\widetilde{u}_+(x) = 2D_1 e^{\sqrt{-P/B}x} \sinh \sqrt{-P/B}\,(x-1) + \widetilde{u}_{+p}(x),$$

where

$$\widetilde{u}_{-p}(x) = \frac{1}{\sqrt{-P/A}} \int_0^x g(\tau) \sinh \sqrt{-P/A}(x - \tau)\, d\tau$$

and similarly to $\widetilde{u}_{+p}(x)$ (replacing P/A with P/B and integration limits from 1 to x). The stability conditions (19-20) are equivalent to the linear system $Hy = z$ with

$$H := \left[\begin{array}{cc} 2A \sinh \sqrt{-P/A}x_0 & -2Be^{\sqrt{-P/B}} \sinh \sqrt{-P/B}(x_0 - 1) \\ 2A\sqrt{-P/A} \cosh \sqrt{-P/A}x_0 & -2B\sqrt{-P/B}e^{\sqrt{-P/B}} \cosh \sqrt{-P/B}(x_0 - 1) \end{array} \right] \tag{21}$$

and

$$y := \left[\begin{array}{c} C_1 \\ D_1 \end{array} \right], \qquad z := \left[\begin{array}{c} Bu_{+p}(x_0) - Au_{-p}(x_0) \\ Bu'_{+p}(x_0) - Au'_{-p}(x_0) \end{array} \right].$$

Further we have

$$\det H = 4ABe^{\sqrt{-P/B}} \left(-\sqrt{-P/B} \sinh \sqrt{-P/A}x_0 \cosh \sqrt{-P/B}(x_0 - 1) \right.$$
$$\left. + \sqrt{-P/A} \sinh \sqrt{-P/B}(x_0 - 1) \cosh \sqrt{-P/A}x_0 \right).$$

Since

$$\sinh \sqrt{-A/P}x_0 \cosh \sqrt{-B/P}(x_0 - 1) > 0$$

and

$$\sinh \sqrt{-B/P}(x_0 - 1) \cosh \sqrt{-A/P}x_0 < 0$$

we have $\det H < 0$ hence unique solvability of the above linear system.

Case $P > 0$: If $P > 0$ then the solutions $\widetilde{u}_-$, $\widetilde{u}_+$ involve sin and cos (instead of sinh and cosh) and the determinant of the corresponding linear system $Hy = z$ reads

$$\det H = 4ABe^{\sqrt{-P/B}} \left(-\sqrt{P/B} \sin \sqrt{P/A}x_0 \cos \sqrt{P/B}(x_0 - 1) \right.$$
$$\left. + \sqrt{P/A} \sin \sqrt{P/B}(x_0 - 1) \cos \sqrt{P/A}x_0 \right).$$

When $\det H \neq 0$ we have the same situation as in the case $P < 0$. Observe that the set Z_0 of values for P such that any cosine factor occurring in the above determinant vanishes is at most countable. Apart from these values, to find $P > 0$ for which $\det H = 0$ is equivalent to solving

$$h(s) := \tan(s) + \nu\mu\tan(\mu s) = 0,$$

where $s = \sqrt{P/A}x_0 > 0$, $\mu = \sqrt{A/B}(1/x_0 - 1) > 0$, and $\nu = x_0/(1 - x_0) > 0$. One observes that there is a countable discrete set of singularities of h, at which the limits from the left and right are $+\infty$ and $-\infty$ respectively. Since h is continuous otherwise, there is a countable set Z_1 of (positive) zeroes. To summarize, the union $Z_0 \cup Z_1$ makes up a sequence $(P_l)_{l\in\mathbb{N}}$ with the required property. ∎

Remark 3.5. We point out that the above proof of Lemma 3.3 does not give the minimum set of values P_l to be removed. In fact, only those elements in Z_0 have to occur in (P_l) which make both cosine factors vanish. Note that the latter can only happen, when $\sqrt{B/A}x_0/(1 - x_0)$ is a rational number of the form $(2l + 1)/(2k + 1)$ with integers k, l.

Theorem 3.6. *Let P satisfy one of the conditions (i)-(ii) in Lemma 3.3. Let $\widetilde{u}_-$ and $\widetilde{u}_+$ be the solutions to (10) and (11) obtained in Lemma 3.3 and define*

$$u_-(x) = \left\{ \begin{array}{ll} \widetilde{u}_-(x) & x \in [0, x_0] \\ 0 & x \in [x_0, 1] \end{array} \right. , \qquad u_+(x) = \left\{ \begin{array}{ll} 0 & x \in [0, x_0] \\ \widetilde{u}_+(x) & x \in [x_0, 1] \end{array} \right. . \tag{22}$$

Then

$$u(x) = u_-(x) + u_+(x) \tag{23}$$

is the unique solution to (7-8) in the sense of Definition 2.1, belongs to $L^1([0, 1])$ and satisfies the boundary conditions in the classical sense.

Proof. Since $a(x) = AH(x_0 - x) + BH(x - x_0)$ we have

$$(u \cdot a)(x) = Au_-(x) + Bu_+(x)$$

which we will differentiate twice. Recall ([11, Chapitre II, §2]) that if a function f is in $C_{\text{abs}}([0, 1]\backslash\{x_0\})$, such that $\lim_{x\to x_0-} f(x) = f(x_0-)$ and $\lim_{x\to x_0+} f(x) = f(x_0+)$ exist, then the distributional derivative $\frac{d}{dx}f$ satisfies

$$\frac{d}{dx}f(x) = f'(x) + \big(f(x_0+) - f(x_0-)\big) \cdot \delta_{x_0}, \tag{24}$$

where f' denotes the (class of) function(s) in $L^1([0, 1])$ equal to the pointwise derivative of f almost everywhere in $[0, 1] \backslash \{x_0\}$.

Therefore

$$\frac{d}{dx}(u \cdot a) = Au'_- + Bu'_+ + \big(Bu(x_0+) - Au(x_0-)\big) \cdot \delta_{x_0}$$

and

$$\frac{d^2}{dx^2}(u \cdot a) = Au''_- + Bu''_+ + \big(Bu'(x_0+) - Au'(x_0-)\big) \cdot \delta_{x_0}$$
$$+ \big(Bu(x_0+) - Au(x_0-)\big) \cdot \delta'_{x_0}. \tag{25}$$

By construction we have that

$$Au''_-(x) = \left\{ \begin{array}{ll} (-Pu + g)(x) & x \in [0, x_0] \\ 0 & x \in [x_0, 1] \end{array} \right. , \quad Bu''_+(x) = \left\{ \begin{array}{ll} 0 & x \in [0, x_0] \\ (-Pu + g)(x) & x \in [x_0, 1] \end{array} \right. .$$

Thus (17) and (18) imply that

$$\frac{d^2}{dx^2}(u \cdot a) = -Pu + g.$$

Note that u is continuous near the boundaries $x = 0$ and $x = 1$, thus the conditions (8) follow by construction.

For uniqueness, we first observe that any solution u has to be in L^1_{loc} by Theorem 2.4. Furthermore, due to Proposition 3.1 it also has to satisfy the stability conditions (17-18). Hence Lemma 3.3 implies uniqueness. ∎

4   Generalization to discontinuous axial force

We extended the analysis of the previous section to investigate solvability of the same type of differential equation

$$\frac{d^2}{dx^2}[a(x) \cdot u(x)] + Pu(x) = g(x)$$

with boundary condition $u(0) = u(1) = 0$, where the force P now is a jump function of the form

$$P = P_1 + H(x - x_0)(P_1 - P_2)$$

with real numbers P_1, P_2.

As with constant P we obtain that any solution to the differential equation (in a sense similar to Definition 2.1) necessarily is a locally integrable function and continuously differentiable off x_0 with a jump at $x = x_0$.

Remark 4.1. Note that condition (A1) in Definition 2.1 implies that the model product $[P \cdot u]$ exists. Therefore we will now require u to be a solution to the differential equation in the sense of Definition 2.1 with (9) replaced by

$$([a \cdot u])'' + [P \cdot u] = g.$$

Theorem 4.2. *(i) Let $u \in \dot{\mathcal{D}}'([0,1])$ be a solution in the sense of Remark 4.1. Then u is a locally integrable function.*

(ii) If $u \in L^1_{loc}$ is a solution then $u \in C^1([0,1]\backslash\{x_0\})$ and u has a jump discontinuity at $x = x_0$. Furthermore, equations (17) and (18) hold.

Proof. *Step 1:* As in the proof of Theorem 2.4 we can set $\widetilde{u}_- = u\big|_{(0,x_0)}$ and $\widetilde{u}_+ = u\big|_{(x_0,1)}$. Then we have

$$A\widetilde{u}_-'' + P_1\widetilde{u}_- = g\big|_{(0,x_0)} \tag{26}$$

$$B\widetilde{u}_+'' + P_2\widetilde{u}_+ = g\big|_{(x_0,1)} \tag{27}$$

Solving these two differential equations with constant coefficients we get

$$\widetilde{u}_- = \widetilde{u}_{-h} + \widetilde{u}_{-p} \quad \text{and} \quad \widetilde{u}_+ = \widetilde{u}_{+h} + \widetilde{u}_{+p} \tag{28}$$

where

$$\widetilde{u}_{-h}(x) = C_1 e^{\sqrt{-P_1/A}x} + C_2 e^{-\sqrt{-P_1/A}x}, \ \widetilde{u}_{+h}(x) = D_1 e^{\sqrt{-P_2/B}x} + D_2 e^{-\sqrt{-P_2/B}x}$$

and

$$\widetilde{u}_{-p}(x) = \frac{1}{2\sqrt{-P_1/A}}\Big(\int_0^x g(\tau)e^{\sqrt{-P_1/A}(x-\tau)}d\tau - \int_0^x g(\tau)e^{-\sqrt{-P_1/A}(x-\tau)}d\tau\Big),$$

with a similar formula for $\widetilde{u}_{+p}(x)$ replacing P_1/A by P_2/B and integration limits from 1 to x. Again, $\widetilde{u}_{-h}, \widetilde{u}_{+h}$ are smooth and $\widetilde{u}_{-p}, \widetilde{u}_{+p}$ are absolutely continuous. Therefore $\widetilde{u}_-$ and $\widetilde{u}_+$ are absolutely continuous on open subintervals $(0, x_0)$ and $(x_0, 1)$. Also, there exist $\widetilde{u}_-(x_{0-}) = \lim_{x \to x_{0-}} \widetilde{u}_-(x)$ and $\widetilde{u}_+(x_{0+}) = \lim_{x \to x_{0+}} \widetilde{u}_+(x)$.

Step 2: Precisely as in Step 2 of the proof of Theorem 2.4 we obtain

$$u = \widetilde{u} + c\delta_{x_0}. \tag{29}$$

Step 3: Equation (29) and $P = P_1 + H(x - x_0)(P_1 - P_2)$ leads to $Pu = P_1\widetilde{u}_- + P_2\widetilde{u}_+ + \frac{c}{2}\delta_{x_0}(P_2 - P_1)$. Since $P_1\widetilde{u}_- + P_2\widetilde{u}_+ \in L^1_{loc}(\mathbb{R})$ we have that $w = \int(g - P_1\widetilde{u}_- - P_2\widetilde{u}_+)$ is absolutely continuous and its primitive function W is C^1. The differential equation $(u \cdot a)'' = g - Pu$ implies

$$u \cdot a = W - \frac{c}{2}(P_2 - P_1)(x - x_0)_+,$$

8

which is absolutely continuous. The same arguments as in Theorem 2.4 yield that $c = 0$ and hence $u = \widetilde{u}$ is locally integrable. This proves (i).

For part (ii) we may reason as in the proof of Proposition 3.1 we get that second part of theorem is valid. ∎

The construction of a solution rests on the following lemma which corresponds to Lemma 3.3.

Lemma 4.3. *For any choice of $A > 0$, $B > 0$, and $0 < x_0 < 1$ there exist one-dimensional submanifolds $\mathcal{M}$ and $\mathcal{N}$ of $\mathbb{R}^2$ such that the following holds:*

(i) *If $P_1 < 0$ and $P_2 < 0$*

or

(ii) *if $P_1 > 0$, $P_2 > 0$ and $(P_1, P_2) \notin \mathcal{M}$*

or

(iii) *if $P_1 > 0$, $P_2 < 0$, and $(P_1, P_2) \notin \mathcal{N}$*

then there is a unique solution to (26) and (27) with $\widetilde{u}_-(0) = 0$ and $\widetilde{u}_+(1) = 0$, which satisfies the stability conditions

$$A\widetilde{u}_-(x_0) = B\widetilde{u}_+(x_0) \tag{30}$$
$$A\widetilde{u}'_-(x_0) = B\widetilde{u}'_+(x_0). \tag{31}$$

Remark 4.4. Similarly as in Remark 3.4 for the cases where (P_1, P_2) belongs to $\mathcal{M}$ or $\mathcal{N}$ the solution is not unique or may fail to exist, explicit investigation of which could be carried out along the lines of the following proof.

Proof. As in the proof of Lemma 3.3 any solution to (26) and (27) is given by (28).

In case $P_1 < 0$ and $P_2 < 0$ the solution formulae (28) with boundary conditions at 0 and 1, and stability conditions (30) and (31) lead to a linear (2x2) system $Hy = z$ with y and z as in Lemma 3.3 and

$$\det H = 4ABe^{\sqrt{-P_2/B}}\left(-\sqrt{-P_2/B}\sinh\sqrt{-P_1/A}x_0\cosh\sqrt{-P_2/B}(x_0-1)\right.$$
$$\left. +\sqrt{-P_1/A}\sinh\sqrt{-P_2/B}(x_0-1)\cosh\sqrt{-P_1/A}x_0\right) < 0.$$

Therefore we have a unique solution.

Case $P_1 > 0$, $P_2 > 0$: $\det H$ now reads

$$\det H = 4ABe^{\sqrt{P_2/B}}\left(-\sqrt{P_2/B}\sin\sqrt{P_1/A}x_0\cos\sqrt{P_2/B}(x_0-1)\right.$$
$$\left. +\sqrt{P_1/A}\sin\sqrt{P_2/B}(x_0-1)\cos\sqrt{P_1/A}x_0\right).$$

Whenever this is nonzero we have a unique solution. To see where it vanishes let $s = \sqrt{P_1/A}x_0$, $t = \sqrt{P_2/B}(x_0 - 1)$, $\nu = x_0/(1 - x_0)$ and analyze the function

$$f(s,t) = \nu t \sin s \cos t + s \sin t \cos s.$$

By direct inspection one deduces that $\mathrm{grad}\, f$ is nonzero when $f = 0$ which yields that the zero set $\mathcal{M}' = \{(s,t) \in \mathbb{R}^2; f(s,t) = 0\}$ is a one-dimensional submanifold of $\mathbb{R}^2$. We set $\mathcal{M} = \{(P_1, P_2) \in \mathbb{R}^2; (s,t) \in \mathcal{M}'\}$.
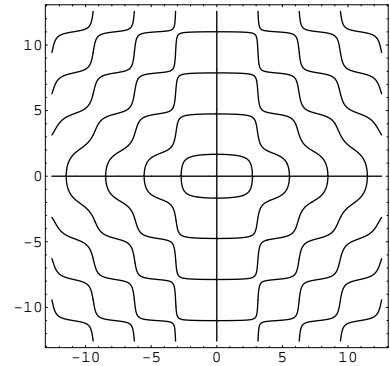

Figure: $\mathcal{M}'$ is a union of infinitely many closed concentric curves (plot for the case $\nu = 6$).

Case $P_1 > 0$, $P_2 < 0$: we have

$$\det H = 4ABe^{\sqrt{P_2/B}} \left(\sqrt{-P_2/B} \sinh \sqrt{-P_1/A}x_0 \cos \sqrt{P_2/B}(x_0 - 1) \right.$$
$$\left. + \sqrt{P_1/A} \sin \sqrt{P_2/B}(x_0 - 1) \cosh \sqrt{P_1/A}x_0 \right).$$

and as above one can show that the zero set is a one-dimensional submanifold $\mathcal{N}$ of $\mathbb{R}^2$. In the complement $\det H \neq 0$, solution exists and is unique. ∎

Theorem 4.5. *Let P_1 and P_2 satisfy one of the conditions (i)-(iii) in Lemma 4.3. Let $\widetilde{u}_-(x)$ and $\widetilde{u}_+(x)$ be the solutions to (26) and (27) obtained in Lemma 4.3 and define*

$$u_-(x) = \begin{cases} \widetilde{u}_-(x) & x \in [0, x_0] \\ 0 & x \in [x_0, 1] \end{cases} \qquad u_+(x) = \begin{cases} 0 & x \in [0, x_0] \\ \widetilde{u}_+(x) & x \in [x_0, 1] \end{cases}. \tag{32}$$

Then

$$u(x) = u_-(x) + u_+(x) \tag{33}$$

is a solution to (7-8) with $P = P_1 + H(x - x_0)(P_1 - P_2)$.

Proof. Proceeding as in the proof of Theorem 3.6 we arrive at (25). By construction we have

$$Au_-'' = \begin{cases} -P_1 u + g & x \in [0, x_0] \\ 0 & x \in [x_0, 1] \end{cases} \qquad \text{and} \qquad Bu_+'' = \begin{cases} 0 & x \in [0, x_0] \\ -P_2 u + g & x \in [x_0, 1] \end{cases}.$$

Employing (30) and (31) we get $\frac{d^2}{dx^2}(u \cdot a) = -(P_1 + (P_2 - P_1)H(x - x_0))u + g$. ∎

5 Approximation by regularization

In this section we investigate the possibility to approximate the solution to (7), (8) using some regularization of the coefficient $a(x)$. Throughout this section we will assume that P is constant and such that the solution to (7), (8) is unique.

Let $Qu := (au)'' + Pu$ and consider the equation $Qu = g \in L^1([0, 1])$. Suppose that a_ε is a smooth regularization of the jump disconuity in a such that $a_\varepsilon \in C^2([0, 1])$ with $\lim_{\varepsilon \to 0} a_\varepsilon^{(j)} = a^{(j)}$ uniformly on compact subsets of $[0, 1] \setminus \{x_0\}$ for derivative orders $j = 0, 1, 2$. Let $Q_\varepsilon u := (au_\varepsilon)'' + Pu$.

Proposition 5.1. *Let $g \in L^1([0, 1])$.*
(i) If $u_\varepsilon \in \dot{\mathcal{D}}'([0, 1])$ denotes the solution to

$$Q_\varepsilon u = g, \qquad u(0) = 0, u(1) = 0,$$

then u_ε belongs to the space $AC^2([0, 1])$ of continuously diifferentiable functions whose derivates are loccaly integrable.
(ii) Let u be the solution to $Qu = g$. Then $u_\varepsilon \to u$ uniformly on compact subsets of $[0, 1] \setminus \{x_0\}$.

Proof. (i): Note that $a_\varepsilon Q_\varepsilon u_\varepsilon = a_\varepsilon g$ is equivalent to the Sturm Liouville problem $Lu_\varepsilon := (pu_\varepsilon')' + qu_\varepsilon = a_\varepsilon g$ with $p := a_\varepsilon^2$ and $q := a_\varepsilon(a_\varepsilon'' + P)$. If $Q_\varepsilon u_\varepsilon = a_\varepsilon g$ then $(a_\varepsilon^2 u_\varepsilon')' = a_\varepsilon g - a_\varepsilon(a_\varepsilon'' + P) \in L^1([0, 1])$ since $a_\varepsilon g \in L^1([0, 1])$ and a_ε is C^2. Therefore $a_\varepsilon^2 u_\varepsilon'$ is absolutely continuous. Since a_ε is bounded from below away from zero this implies that u_ε' is absolutely continuous as well. Thus $u_\varepsilon \in AC^2([0, 1])$.
(ii): Let $v_\varepsilon := u_\varepsilon - u$. Then we obtain

$$Q_\varepsilon v_\varepsilon = ([(a_\varepsilon - a) \cdot u])'' := f_\varepsilon, \qquad v_\varepsilon(0) = v_\varepsilon(1) = 0. \tag{34}$$

By assumption and using that $a_\varepsilon - a$ is C^2 off x_0 we have that $f_\varepsilon \to 0$ in $L^1_{\mathrm{loc}}([0,1] \setminus \{x_0\})$. Integrating twice in (34) gives

$$(a_\varepsilon v_\varepsilon)(x) = \int_0^x (x-y) f_\varepsilon(y) dy + P \int_0^x (x-y) v_\varepsilon dy + a_\varepsilon(0) v_\varepsilon(0) + (a_\varepsilon v_\varepsilon)'(0)\, x. \qquad (35)$$

Let K be a compact subset of $[0, x_0)$. Since u and u_ε are AC^2 on K (as noted in Remark 3.2 and the prooof above) and both belong to $\dot{\mathcal{D}}'([0,1])$ we have that $v_\varepsilon(0) = v'_\varepsilon(0) = 0$. Furthermore, by $|(a_\varepsilon v_\varepsilon)(x)| \geq A|v_\varepsilon| > 0$ we obtain

$$|v_\varepsilon(x)| \leq \frac{1+P}{A} \int_0^x |x-y||f_\varepsilon(y)| dy + \frac{1+P}{A} \int_0^x |x-y||v_\varepsilon(y)| dy.$$
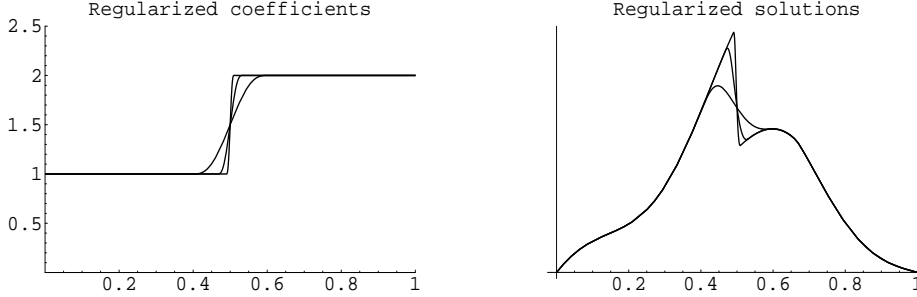
Applying Gronwall's inequality we get

$$|v_\varepsilon(x)| \leq \frac{1+P}{A} e^{\frac{(1+P)x}{A}} \int_0^x |x-y||f_\varepsilon(y)| dy \leq \frac{1+P}{A} e^{\frac{1+P}{A}} \|f_\varepsilon\|_{L^1(K)}.$$

As noted above, we have $f_\varepsilon \to 0$ in $L^1_{\mathrm{loc}}([0,1] \setminus \{x_0\})$, thus $v_\varepsilon \to 0$ uniformly on K.

The reasoning in case of a compact subset contained in $(x_0, 1]$ is similar. Since an arbitrary compact subset of $[0,1] \setminus \{x_0\}$ is the disjoint union of two compact subsets in either part of $[0,1] \setminus \{x_0\}$ the assertion is proved. ∎

We illustrate the convergence in an example: we put $A = 1$, $B = 2$, $P = 1$, $x_0 = 1/2$, and design a_ε as the C^2 function, which is defined by a fifth order odd polynomial in $[-\varepsilon, \varepsilon]$ and equal to a otherwise. As right-hand side we choose $g(x) = -\cos(11x)/\sqrt{|x-2/3|}$. Note that $g \in L^1([0,1]) \setminus L^2([0,1])$. The following plots show the regularized coefficients and corresponding solutions for parameter values $\varepsilon = 1/10, 1/30, 1/100$.

A   Appendix: Hierarchy of distributional products

For convenience of the reader we briefly review the basic definitions of the coherent distributional products described in [10, Chapter II] in terms of a hierarchy. All these products yield the classical multiplication when restricted to smooth functions.

We use Ω to denote an open subset of $\mathbb{R}^n$ and $\widehat{u}$ for the Fourier transform of u.

The most elementary product in this context is $\mathcal{C}^\infty \cdot \mathcal{D}'$, the product of a smooth function and a distribution, defined as the adjoint of multiplication by a smooth function in the test function space.

Disjoint singular support: Assume that u, v are in $\mathcal{D}'(\Omega)$ with disjoint singular supports. Then for any $x \in \Omega$ there is a neighborhood Ω_x and a function $f_x \in \mathcal{D}(\Omega_x)$ such that either $f_x u$ or $f_x v$ is smooth. Then in Ω_x the product of u and v can be defined in the sense of $\mathcal{C}^\infty \cdot \mathcal{D}'$ and by the localization properties of $\mathcal{D}'$ (cf. [3], subsect. 2.2) this consistently defines a distribution in Ω.

Wave front set condition: Let $u \in \mathcal{D}'(\Omega)$ and $(x_0, \xi_0) \in T^*\Omega \setminus 0 := \{(x, \xi) \mid x \in \Omega, \xi \neq 0\}$ (the cotangent bundle over Ω with the zero section removed). u is said to be *microlocally regular* at (x_0, ξ_0) if there is $\varphi \in \mathcal{D}(\Omega)$, $\varphi(x_0) \neq 0$, and an open cone Γ with axial vector ξ_0 such that $\widehat{\varphi u}$ is rapidly decreasing in Γ. $\mathrm{WF}(u)$ is the closed subset of $T^*\Omega \setminus 0$ where u is not microlocally regular.

If u, $v \in \mathcal{D}'(\Omega)$ their wave front sets are said to be in *favorable position* if $(x, \xi) \in \mathrm{WF}(u)$ implies that $(x, -\xi) \notin \mathrm{WF}(v)$. In this case the product of u and v can be defined as the pullback of the tensor product $u \otimes v \in \mathcal{D}'(\Omega \times \Omega)$ by the diagonal map $\Omega \to \Omega \times \Omega$, $x \mapsto (x, x)$ (cf. [3], Thm. 8.2.10).

Fourier product: Given two distributions $u, v \in \mathcal{D}'(\Omega)$ we say that their Fourier product exists if for every $x \in \Omega$ there is an open neighborhood Ω_x and $f_x \in \mathcal{D}(\Omega)$, $f_x = 1$ on Ω_x, such that the $\mathscr{S}'$-convolution of $\widehat{f_x u}$ and $\widehat{f_x v}$ exists. Locally near x, the product of u and v is then defined to be the inverse Fourier transform of $\widehat{f_x u} * \widehat{f_x v}$ (for a definition of $\mathscr{S}'$-convolvability see [10], sect. 6).

Duality products: Let X be a *normal space of distributions*, that is $\mathcal{D} \subseteq X \subseteq \mathcal{D}'$ and $\mathcal{D}$ is dense in X. Assume that the dual space X' is (equipped with a locally convex topology so that it becomes) normal as well and that multiplication with a fixed element in $\mathcal{D}$ induces a continuous linear map both from X into X and from X' into X'.

For any normal space of distributions Y denote by Y_{loc} the set of distributions $v \in \mathcal{D}'$ such that $\psi v \in Y$ for all $\psi \in \mathcal{D}$. If $u \in (X')_{\mathrm{loc}}$ and $v \in X_{\mathrm{loc}}$ then the product of u and v can be defined by

$$\langle u \cdot v, \psi \rangle := \langle \chi u, \psi v \rangle$$

for $\psi \in \mathcal{D}$ and $\chi \in \mathcal{D}$ chosen arbitrarily with $\chi = 1$ on $\mathrm{supp}(\psi)$. Note that in the above definition the left hand side denotes a $(\mathcal{D}', \mathcal{D})$ pairing while the right hand side uses the pairing (X', X).

Strict and model products: The basic idea is to regularize one or both factors by convolution, perform the multiplication in the sense $\mathcal{C}^\infty \cdot \mathcal{D}'$ or $\mathcal{C}^\infty \cdot \mathcal{C}^\infty$, and try to take the limit. The regularizing convolutions are carried out with two principal types of mollifiers.

A net $(\rho^\varepsilon)_{\varepsilon > 0}$ in $\mathcal{D}(\mathbb{R}^n)$ is called *strict delta net* if

$$\mathrm{supp}(\rho^\varepsilon) \to \{0\} \quad \text{as } \varepsilon \to 0 \tag{36}$$

$$\int \rho^\varepsilon(x)\, dx = 1 \quad \text{for all } \varepsilon > 0 \tag{37}$$

$$\int |\rho^\varepsilon(x)|\, dx \quad \text{is bounded independently of } \varepsilon. \tag{38}$$

A *model delta net* is given by specifying $\varphi \in \mathcal{D}(\mathbb{R}^n)$ with $\int \varphi(x)\, dx = 1$ and defining $(\varphi_\varepsilon)_{\varepsilon > 0}$ by $\varphi_\varepsilon(x) = \varphi(x/\varepsilon)/\varepsilon^n$.

Consider the following four possibilities to define a product of u and v:

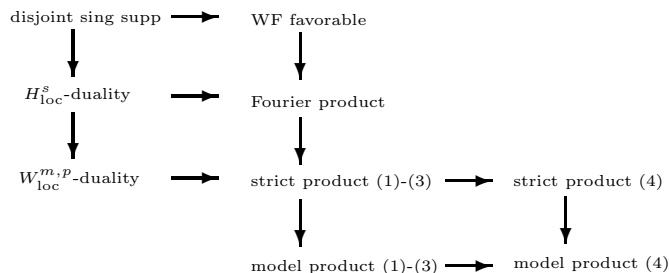$$u \cdot [v] = \lim_{\varepsilon \to 0} u(v * \rho^\varepsilon) \tag{1}$$

$$[u] \cdot v = \lim_{\varepsilon \to 0} (u * \rho^\varepsilon) v \tag{2}$$

$$[u] \cdot [v] = \lim_{\varepsilon \to 0} (u * \rho^\varepsilon)(v * \sigma^\varepsilon) \tag{3}$$

$$[u \cdot v] = \lim_{\varepsilon \to 0} (u * \rho^\varepsilon)(v * \rho^\varepsilon) \tag{4}$$

where the limit is required to exist in $\mathcal{D}'(R^n)$ and independent of the choice of $(\rho^\varepsilon)_{\varepsilon > 0}$ and $(\sigma^\varepsilon)_{\varepsilon > 0}$ in the class of strict, resp. model, delta nets. This defines four types of so-called *strict*, resp. *model*, products. Since the definitions (1)-(3) turn out to be equivalent when using strict, resp. model, delta nets (cf. [10], Thms. 7.2 and 7.11) we distinguish only the following four products: strict product (1)-(3), strict product (4), model product (1)-(3), and model product (4).

Coherence properties: The various products satisfy coherence properties and can be brought into the following hierarchy table. Here, an arrow indicates that a product definition is contained and consistent with its successor in the graph. All products shown generalize the multiplication $\mathcal{C}^\infty \cdot \mathcal{D}'$.

$$
\begin{array}{ccccc}
\text{disjoint sing supp} & \longrightarrow & \text{WF favorable} & & \\
\downarrow & & \downarrow & & \\
H^s_{\text{loc}}\text{-duality} & \longrightarrow & \text{Fourier product} & & \\
\downarrow & & \downarrow & & \\
W^{m,p}_{\text{loc}}\text{-duality} & \longrightarrow & \text{strict product (1)-(3)} & \longrightarrow & \text{strict product (4)} \\
& & \downarrow & & \downarrow \\
& & \text{model product (1)-(3)} & \longrightarrow & \text{model product (4)}
\end{array}
$$

References

[1] T. M. Atanackovic. *Stability theory of Elastic Rods.* World Scientific, 1997.

[2] M. Grosser, M. Kunzinger, M. Oberguggenberger, and R. Steinbauer. *Geometric theory of generalized functions.* Kluwer, Dordrecht, 2001.

[3] L. Hörmander. *The analysis of linear partial differential operators*, volume I. Springer-Verlag, second edition, 1990.

[4] G. Hörmann. Zygmund classes in algebras of generalized functions. *Zeitschr. f. Anal. Anw.*, 23:139–165, 2004.

[5] G. Hörmann and M. V. de Hoop. Microlocal analysis and global solutions of some hyperbolic equations with discontinuous coefficients. *Acta Appl. Math.*, 67:173–224, 2001.

[6] G. Hörmann and M. V. de Hoop. Geophysical modeling with Colombeau functions: Microlocal properties and Zygmund regularity. In A. Delcroix, M. Hasler, J.-A. Marti, and V. Valmorin, editors, *Nonlinear Algebraic Analysis.* Cambridge Scientific Publishers, 2004.

[7] F. Lafon and M. Oberguggenberger. Generalized solutions to symmetric hyperbolic systems with discontinuous coefficients: the multidimensional case. *J. Math. Anal. Appl.*, 160:93–106, 1991.

[8] M. Oberguggenberger. Hyperbolic systems with discontinuous coefficients: examples. In B. Stanković, E. Pap, S. Pilipović, and V. S. Vladimirov, editors, *Generalized Functions, Convergence Structures, and Their Applications*, pages 257–266, New York, 1988. Plenum Press.

[9] M. Oberguggenberger. Hyperbolic systems with discontinuous coefficients: generalized solutions and a transmission problem in acoustics. *J. Math. Anal. Appl.*, 142:452–467, 1989.

[10] M. Oberguggenberger. *Multiplication of distributions and applications to partial differential equations.* Longman Scientific & Technical, 1992.

[11] L. Schwartz. *Théorie des Distributions.* Hermann, Paris, 1966.

[12] A. Yavari and S. Sarkani. On applications of generalized functions to the analysis of euler-bernuli beam columns with jump discontinuities. *Intern. Jour. Mech. Sci.*, 43:1543–1562, 2001.